proprep™

Workbook

Singular Points

Zeros of Analytic Functions

Questions

1) Find the order of the zero of $f(z) = z\sin z$ at $z_0 = 0$.

2) Find the order of the zero of $f(z) = z\sin z^3$ at $z_0 = 0$.

3) Suppose that $f_1(z)$ is analytic at z_0 and has a zero of order m_1 there and suppose, too, that $f_2(z)$ is analytic at z_0 and has a zero of order m_2 there. Prove that $f(z) = f_1(z) f_2(z)$ is analytic at z_0 and has a zero of order $m = m_1 + m_2$ there.

4) Find the order of the zero of $f(z) = z^{20}\sin z$ at $z_0 = 0$.

5) Find the order *m* of the zero of $f(z) = e^{\sin z} - \sin^2 z - 1$ at $z_0 = 0$.

6) Suppose that $f_1(z)$ is analytic at z_0 and has a zero of order 7 there and suppose, too, that $f_2(z)$ is analytic at z_0 and has a zero of order 3 there. Show that z_0 is a zero of $f(z) = f_1(z) + f_2(z)$ and find its order.

7) Find the order *m* of the zero of $f(z) = 6\sin z^3 + z^{12}(z^6 - 6)$ at $z_0 = 0$.

8) Let $f(z)$ be continuous on the unit disk $D = D(0,1)$ and assume $f(z) \not\equiv 0$. Suppose that $f^3(z),\ f^7(z)$ are both holomorphic on *D*. Prove that *f* is holomorphic on D.

9) Prove: there is no holomorphic function $f(z)$ on the unit disk $D = D(0,1)$ such that $f\left(\frac{1}{n}\right) = \frac{1}{\sqrt{n}}$ for $n = 2, 3, 4, \ldots$

Classifying Singular Points

10) Find the radius of convergence of the series $\sum_{n=0}^{\infty} e^{in} z^n$.

11) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \left(\frac{z}{in} \right)^n$.

12) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{1}{3^n (2n+1)} (z-3)^n$.

Singular Points at Infinity

13) Classify the singularity at ∞ of $f(z) = \frac{z^2}{1+z}$.

14) Classify the singularity at ∞ of $f(z) = e^z$.

Casorati–Weierstrass Theorem

15) A) Prove that $z = i$ is an essential singularity of $f(z) = \cos\left(\frac{1}{z^2+1} \right)$.

b) Deduce that there exists $z \in \mathbb{C}$ such that

$$\left| \cos\left(\frac{1}{z^2+1} \right) + 100 \tan^2 z + e^{-z^2} - 5i \right| < 1$$

Answer Key :

1) order 2

2) order 4

3) (proof)

4) order 21

5) $m = 1$

6) order 3

7) $m = 3$

8) (proof)

9) (proof)

10) $R = 1$

11) $R = \infty$

12) $R = 3$

13) pole of order 1 [simple pole]

14) essential singularity

15) [proof]