proprep™

Workbook

Sequences and Topology

Sequences and Topology

Questions

1)

$$z_n = \frac{1}{n} + i\left(\frac{n-2}{n}\right)$$

$$\lim_{n\to\infty} z_n = ?$$

2)

$$z_n = \frac{n^2+n+1}{3n^2+2} + i\left(\frac{n-1}{2n}\right)$$

$$\lim_{n\to\infty} z_n = ?$$

3)

Given: $z_n = i^{2n} \cdot n^3$

Prove: $\lim_{n\to\infty} |z_n| = \infty$ and use this to evaluate $\lim_{n\to\infty} z_n$.

4)

Given: $z_n = \dfrac{i^n}{n}$

Prove: $\lim_{n\to\infty} |z_n| = 0$ and use this to evaluate $\lim_{n\to\infty} z_n$.

5)

Given: $z_n = \dfrac{(1+i)^n}{n}$

Prove: $\lim\limits_{n\to\infty} |z_n| = \infty$ and use this to evaluate $\lim\limits_{n\to\infty} z_n$.

6)

Given a sequence $z_n = n \cdot z^n$ where $z \in \mathbb{C}$ is fixed.

Find $\lim\limits_{n\to\infty} |z_n|$ and $\lim\limits_{n\to\infty} z_n$ for two cases:

a) $|z| \geq 1$ b) $0 < |z| < 1$

7)

Prove: If $z_n \xrightarrow{n\to\infty} z$ and $w_n \xrightarrow{n\to\infty} = w$

then $z_n + w_n \xrightarrow{n\to\infty} z + w$

8)

For each sequence, check if it converges; if so, compute its limit:

a) $z_n = \dfrac{1+n}{1-2n} + \dfrac{n-10}{n^2} i$

b) $z_n = \cos(\pi n) + n \sin\left(\dfrac{1}{n}\right) i$

c) $z_n = \left(1 + \dfrac{2}{n}\right)^{-n} + \sqrt[n]{3^n + 4^n} \cdot i$

d) $z_n = \left(\dfrac{1+\sqrt{3}i}{2}\right)^n$

9)

Sketch the set *S* defined by the inequality $|z-i|+|z+i|<4$.

a) Is *S* open?

b) Is *S* a domain?

c) Is *S* a simply-connected domain

10)

Sketch the set *S* defined by the equation $\mathrm{Re}\left[\frac{z-a}{z+a}\right]=0$, where $a\neq 0$ is real.

a) Is *S* open?

b) Is *S* a domain?

c) Is *S* a simply-connected domain?

11)

Sketch the set *S* defined by the equation $\mathrm{Im}\left[\frac{z-1}{z+1}\right]=0$.

a) Is *S* open?

b) Is *S* a domain?

c) Is *S* a simply-connected domain?

12)

Sketch the set *S* defined by the equation $|z+i|=2|z-i|$.

a) Is *S* open?

b) Is *S* a domain?

c) Is *S* a simply-connected domain?

13)

Show that for any $z, w \in \mathbb{C}$,

$|z \pm w|^2 = |z|^2 \pm 2\operatorname{Re}\left(z \cdot \overline{w}\right) + |w|^2$.

Let $z_1, z_2 \in \mathbb{C}$ and $k > 0$ (real).

Show that the equation $|z - z_1| = k \cdot |z - z_2|$

describes a line $(k = 1)$ or a circle $(k \neq 1)$.

Answer Key

1)

$$\lim_{n\to\infty} z_n = i$$

2)

$$\lim_{n\to\infty} z_n = \frac{1}{3} + i\frac{1}{2}$$

3)

$$\lim_{n\to\infty} z_n = \infty$$

4)

$$\lim_{n\to\infty} z_n = 0$$

5)

$$\lim_{n\to\infty} z_n = \infty$$

6)

a) $\lim_{n\to\infty} z_n = \infty$

b) $\lim_{n\to\infty} z_n = 0$

7)

(Proof)

8)

a) Converges to $-\frac{1}{2}$.

b) Doesn't converge.

c) Converges to $\frac{1}{e^2}+4i$.

d) Doesn't converge.

9)

Yes

Yes

Yes

10)

a. *S* is not open.

b. Hence *S* is not a domain.

c. Hence *S* is not a simply-connected domain.

11)

a. *S* is not open.

b. Hence *S* is not a domain.

c. Hence *S* is not a simply-connected domain.

12)

a. *S* is not open.

b. Hence *S* is not a domain.

c. Hence *S* is not a simply-connected domain.

13)

(Proof)