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# Workbook

# Schwarz's Lemma

## Schwarz's Lemma

### Questions

**1)** Let $f, g : D(0,1) \to D(0,1)$ be holomorphic equivalences and suppose that,

for some $z_1 \in D(0,1)$, $f(z_1) = 0$ and $g(0) = z_1$. $\text{Reminder: } D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

Show that $|f(z)| \le |g^{-1}(z)|$ for all $z \in D(0,1)$.

**2)** Let $f, g : D(0,1) \to D(0,1)$ be holomorphic equivalences and suppose that,

for some $z_1 \in D(0,1)$, $f(z_1) = g(z_1) = 0$.

a) Show that $|f(z)| \le |g(z)|$ for all $z \in D(0,1)$.

b) Deduce from the above that $|f(z)| = |g(z)|$ for all $z \in D(0,1)$.

**3)** Let $g$ be the Möbius Transformation $g(z) = \dfrac{z-a}{1-\bar{a}z}$ where $a \in D(0,1)$.

a) Show that if $|z| = 1$ then $|g(z)| = 1$; i.e., $g$ maps the unit circle to itself.

b) Deduce from the above that $g$ maps $D(0,1)$ to itself: $g : D(0,1) \to D(0,1)$.

c) Let $f : D(0,1) \to D(0,1)$ be a holomorphic equivalence which satisfies $f(a) = 0$.

Prove that, for all $z \in D(0,1)$, $|f(z)| = \left|\dfrac{z-a}{1-\bar{a}z}\right|$.[Hint: use a previous exercise.]

**4)** Let $g$ be the Möbius Transformation $g(z) = \dfrac{z-a}{z-\bar{a}}$, where $a \in H^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

a) Show that $g$ maps the real axis to the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

b) Deduce from the above that $g$ maps $H^+$ to $D(0,1)$.

c) Let $f : H^+ \to D(0,1)$ be a holomorphic equivalence which satisfies $f(a) = 0$.

Prove that $|f(z)| \le \left|\dfrac{z-a}{z-\bar{a}}\right|$ for all $z \in H^+$.

**5)** Prove the following generalization of Schwarz's Lemma.

Suppose that:

a) $f(z)$ is holomorphic in $D(0,1)$.

b) For all $z \in D(0,1)$, $|f(z)| \le 1$

c) $f(0) = f^{(1)}(0) = \ldots = f^{(m-1)}(0) = 0$ for some natural number $m \ge 1$.

Then:

d) For all $z \in D(0,1)$, $|f(z)| \le |z|^m$

**6)** Denote $D = \{z \in \mathbb{C} : |z| < 1\}$ and $H^r = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

a) Define the Möbius Transformation $\varphi(z) = \dfrac{z-\alpha}{z+\bar{\alpha}}$, where $\alpha \in H^r$, arbitrary.

Show that $\varphi(H^r) = D$.

b) Let $f : H^r \to D$ be analytic and suppose that $f(\alpha) = f'(\alpha) = 0$ for some $\alpha \in H^r$.

Prove that $|f(w)| \le \left|\dfrac{w-\alpha}{w+\bar{\alpha}}\right|^2$ for all $w \in H^r$.

c) Let $f : H^r \to D$ be analytic and suppose $f(\alpha) = f(\beta) = 0$, where $\alpha, \beta \in H^r$ and $\alpha \ne \beta$.

Prove that $|f(w)| \le \left|\dfrac{w-\alpha}{w+\bar{\alpha}}\right| \cdot \left|\dfrac{w-\beta}{w+\bar{\beta}}\right|$ for all $w \in H^r$.

**7)** Denote $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f : D \to D$ be analytic.

a) Prove that for all $z_1, z_2 \in D$, $\left|\frac{f(z_2)-f(z_1)}{1-\overline{f(z_2)}f(z_1)}\right| \le \left|\frac{z_2-z_1}{1-\overline{z_2}z_1}\right|$

b) Prove that for all $z \in D$, $\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$

Hint: use the mapping $\varphi_a : D \to D,\ \varphi_a(z) = \frac{z-a}{1-\bar{a}z}$.

**8)** Denote $D = \{w \in \mathbb{C} : |w| < 1\}$ and $H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Let $f : H^+ \to H^+$ be analytic and let $z_0 \in H^+$.

Prove that $\left|\frac{f(z)-f(z_0)}{f(z)-\overline{f(z_0)}}\right| \le \left|\frac{z-z_0}{z-\overline{z_0}}\right|,\ \forall z \in H^+$

Hint: use the mapping $\varphi_a : H^+ \to D,\ \varphi_a(z) = \frac{z-a}{z-\bar{a}},\ a \in H^+$.

**9)** Let $f : D(0,1) \to D(0,1)$ be analytic and let $\alpha, \beta \in D(0,1)$ be two different fixed points of $f$.

In other words, $\alpha \ne \beta,\ f(\alpha) = \alpha,\ f(\beta) = \beta$. Prove that $f(z) \equiv z$.

**10)** Let $f : D(0,1) \to D(0,1)$ be analytic. Prove that $|f'(z)| \le \frac{1}{1-|z|^2}$ for all $z \in D(0,1)$.

**All questions here are proofs, so there are no "answers".**