

Workbook



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Rolle`s Theorem and the Mean Value Theorem

The Mean Value Theorem

Questions:

- 1) Prove that $\frac{b-a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b-a}{a}$ ($0 < a < b$).
- 2) Prove that $\frac{b-a}{\cos^2 a} < \tan b - \tan a < \frac{b-a}{\cos^2 b}$ ($0 < a < b < \frac{\pi}{2}$).
- 3) Prove that $(a-b)e^{-a} < e^{-b} - e^{-a} < (a-b)e^{-b}$ ($a < b$).
- 4) Prove that $\frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-1}{1+a^2}$ ($0 > a > b$).
- 5) Prove that $\frac{b-a}{\sqrt{1-a^2}} < \arcsin b - \arcsin a < \frac{b-a}{\sqrt{1-b^2}}$ ($0 < a < b < 1$).
- 6) Prove that $\frac{b-a}{\sqrt{1+b^2}} < \frac{\operatorname{arcsinh}(b) - \operatorname{arcsinh}(a)}{b-a} < \frac{b-a}{\sqrt{1+a^2}}$ ($0 < a < b$).
- 7) Prove that $\frac{b-a}{1-a^2} < \operatorname{arctanh}(b) - \operatorname{arctanh}(a) < \frac{b-a}{1-b^2}$ ($0 < a < b < 1$).
- 8) Prove that $\sqrt[n]{b} \cdot \frac{b-a}{n \cdot b} < \sqrt[n]{b} - \sqrt[n]{a} < \sqrt[n]{a} \cdot \frac{b-a}{n \cdot a}$ ($0 < a < b$).
- 9) Prove that $\frac{2b(b-a)}{b^2+1} < \ln\left(\frac{b^2+1}{a^2+1}\right) < \frac{2a(b-a)}{a^2+1}$ ($0 < a < b$).

- 10) Prove that $x < \tan x < \frac{x}{\cos^2 x}$ $\left(0 < x < \frac{\pi}{2}\right)$.
- 11) Prove that $\frac{x}{1+x^2} < \arctan x < x$ $(x > 0)$.
- 12) Prove that $\frac{x}{\sqrt{1+x^2}} < \operatorname{arcsinh}(x) < x$ $(x > 0)$.
- 13) Prove that $x < \operatorname{arctanh}(x) < \frac{x}{1-x^2}$ $(0 < x < 1)$.
- 14) Prove that $1+x < e^x < 1+xe^x$ $(x > 0)$.
- 15) Prove that $\sin x \leq x$ $(x > 0)$.
- 16) Prove that $\tan x < 4x$ $\left(0 < x < \frac{\pi}{3}\right)$.
- 17) Prove that $\arctan x > \ln(1+x)$ $(0 < x < 1)$.
- 18) Prove that $|\sin x_2 - \sin x_1| \leq |x_2 - x_1|$.
- 19) Prove that $|\cos x_2 - \cos x_1| \leq |x_2 - x_1|$.
- 20) Prove that $|\arctan y - \arctan x| < |y - x|$.
- 21) Prove that $|\tan y - \tan x| \leq 8|\sin y - \sin x|$, $x, y \in \left[0, \frac{\pi}{3}\right]$.
- 22) Prove that $\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}$.
- 23) Prove that $\frac{1}{2\sqrt{2}} + 1 < \sqrt{2} < 1.5$.

24) Prove that $\frac{3}{25} + \frac{\pi}{4} < \arctan\left(\frac{4}{3}\right) < \frac{1}{6} + \frac{\pi}{4}$.

25) Prove that $\frac{\sqrt{3}}{15} + \frac{\pi}{6} < \arcsin(0.6) < \frac{1}{8} + \frac{\pi}{6}$.

26) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \leq 5$.

It is known that $f(1) = 3$, $f(4) = 18$.

Prove that $f(2) = 8$.

27) Let $f(x)$ be a function which is differentiable for all x and satisfies $|f'(x)| \leq 7$.

It is known that $f(1) = 3$, $f(4) = 18$.

Prove that $4 \leq f(2) \leq 10$.

Answer Key

Solutions in the recordings.

Rolle's Theorem

Example Questions

- 1) Check if the given function f satisfies the conditions of Rolle's Theorem on the given interval. If so, find all the values of c as in the conclusion of Rolle's Theorem.
 - a. $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$.
 - b. $f(x) = \frac{x^2 - 1}{x - 2}$ on $[-1, 1]$.

- 2) Given: $f(x) = \frac{1}{(x-3)^2}$. Show that $f(1) = f(5)$, but that there is no value c in $1 < c < 5$ such that $f'(c) = 0$. Does this contradict Rolle's Theorem?

- 3) Prove that the equation $x^2 + x^3 + 5x = 1$ has exactly one solution.
Hint: show at least one solution and at most one solution.

Answer Key

- 1) a. $f(0) = 0, f(2) = 0$; Yes, it is continuous and differentiable
$$c = \frac{3 \pm \sqrt{3}}{3} \cong 1.6 \text{ and } -0.4$$
- b. $f(-1) = 0, f(1) = 0$; Yes, it is continuous and differentiable
$$c = 2 - \sqrt{3}$$

2-3) Solutions in the recordings.