

Workbook



Table of Contents

Systems of Linear ODEs.....	2
Linear Algebra - Eigenvalues and Eigenvectors	2
1 st Order Homogeneous with Constant Coefficients – Diagonalization	4
1 st Order Nonhomogeneous with Constant Coefficients - Variation of Parameters.....	6
The Substitution Method	7



Systems of Linear ODEs

Linear Algebra - Eigenvalues and Eigenvectors

Questions

Find the eigenvalues and eigenvectors of the matrix:

$$1) \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$2) \quad A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{pmatrix}$$

$$3) \quad A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$4) \quad A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$5) \quad A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$6) \quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

$$7) \quad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

$$8) \quad A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Answer Key

- 1) $x=0, x=1, x=2, v_{x=0} = (-1, 0, 1), v_{x=1} = (0, 1, 0), v_{x=2} = (1, 0, 1)$
- 2) $x=6, x=2, x=-4, v_{x=6} = (0, 0, 1), v_{x=2} = (1, 1, 1), v_{x=-4} = (-1, 1, 0)$
- 3) $x_1 = 2, x_2 = 3, v_{x=2} = (1, 1, 1), v_{x=3}^{(1)} = (1, 0, 1), v_{x=3}^{(2)} = (1, 1, 0)$
- 4) $x=1, x=3, x=-2, v_{x=1} = (-1, 4, 1), v_{x=1} = (1, 2, 1), v_{x=-2} = (-1, 1, 1)$
- 5) $x=1, x=4, x=-1, v_{x=1} = (1, -2, 1), v_{x=4} = (1, 1, 1), v_{x=4} = (1, 1, 1)$
- 6) $x=-1, x=3, v_{x=-1} = (-1, 2), v_{x=3} = (1, 2)$
- 7) $x=1+2i, v_{x=1+2i} = (1+i, 2)$
- 8) $x=1, x=1+\sqrt{3}i, x=1-\sqrt{3}i$
 $v_{x=1} = (1, 1, 1), v_{x=1+\sqrt{3}i} = (1-\sqrt{3}i, 1+\sqrt{3}i, -2), v_{x=1-\sqrt{3}i} = (1+\sqrt{3}i, 1-\sqrt{3}i, -2)$



1st Order Homogeneous with Constant Coefficients – Diagonalization

Questions

Solve the following systems:

$$1) \quad \underline{x}'(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \underline{x}(t)$$

$$2) \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad \underline{x}(0) = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$$

$$3) \quad \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}}_A \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}; \quad \underline{x}(0) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}. \text{ Prove that } z(t) = y(t).$$

Solve the following systems of equations:

$$4) \quad \begin{aligned} x' &= x - y + 4z \\ y' &= 3x + 2y - z \\ z' &= 2x + y - z \end{aligned}$$

$$5) \quad \underline{x}'(t) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \underline{x}(t)$$

$$6) \quad \text{Given the system: } \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}; \quad \underline{x}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

$$\text{Compute: } \lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} + \lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)}.$$

Solve the following systems of equations:

$$7) \quad \begin{cases} y_1' + 5y_1 - 2y_2' = 0 \\ 3y_2' - 4y_1' - 5y_2 = 0 \end{cases}$$

$$8) \quad \bar{x}'(t) = A \cdot \bar{x}(t) \text{ where } A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Answer Key

1) $x_1(t) = -c_1 + c_3 e^{2t}$, $x_2(t) = c_2 e^t$, $x_3(t) = c_1 + c_3 e^{2t}$

2) $\underline{x}(t) = e^{6t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 2e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3e^{-4t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

3) Solution in the recording.

4) $\underline{x}(t) = c_1 e^{1t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

5) $\underline{x}(t) = c_1 e^{1t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

6) $\lim_{t \rightarrow \infty} \frac{y(t)}{x(t)} + \lim_{t \rightarrow -\infty} \frac{y(t)}{x(t)} = 0$

7) $\underline{x}(t) = c_1 e^t \left[\cos 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + c_2 e^t \left[\cos 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$

8) $\underline{x}(t) = c_1 e^{1t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^t \left[\cos \sqrt{3}t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - \sin \sqrt{3}t \begin{pmatrix} -\sqrt{3} \\ \sqrt{3} \\ 0 \end{pmatrix} \right] + c_3 e^t \left[\sin \sqrt{3}t \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \cos \sqrt{3}t \begin{pmatrix} -\sqrt{3} \\ \sqrt{3} \\ 0 \end{pmatrix} \right]$

1st Order Nonhomogeneous with Constant Coefficients - Variation of Parameters

Questions

Solve the following equations:

$$1) \quad \begin{aligned} x_1' &= x_1 + x_2 + 2e^{-t} \\ x_2' &= 4x_1 + x_2 + 4e^{-t} \end{aligned}$$

$$2) \quad \begin{aligned} x_1' &= x_1 + x_2 + e^{at} \\ x_2' &= 4x_1 + x_2 - 2e^{at} \end{aligned}$$

$$3) \quad \underline{x}'(t) = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 18t \\ 3 \\ 0 \end{pmatrix}$$

$$4) \quad \begin{aligned} x' &= x + y + 2z + e^t \\ y' &= x + 2y + z \\ z' &= 2x + y + z + e^t \end{aligned}$$

5) Convert the equation $y''' + y'' - 2y = t^2$ to a system of first order linear ODE (no need to solve).

Answer Key

$$1) \quad \underline{x}(t) = \underbrace{c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{x_h(t)} - \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{x_p(t)}$$

$$2) \quad \underline{x}(t) = \begin{cases} c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} t e^{-t} \\ -2e^{-t} \end{pmatrix} & ; a = -1 \\ c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{a+1} \begin{pmatrix} e^{at} \\ -2e^{at} \end{pmatrix} & ; a \neq -1 \end{cases}$$

$$3) \quad \underline{x}(t) = c_1 e^t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + (3t+2) \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} - (3t+1) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + (-3t+1) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$4) \quad \begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \left(\frac{1}{3} t e^t\right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \left(-\frac{2}{9} e^t\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$5) \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ t^2 \end{pmatrix}$$

The Substitution Method

Questions

Solve the following systems:

$$1) \begin{cases} y'' + 2z' = e^{3x} & (1) \\ y' - z'' + 3z = x^2 & (2) \end{cases}$$

$$2) \begin{cases} y'' + z' = e^{-2x} & (1) \\ y + z = \sin x & (2) \end{cases} \quad z(0) = y(0) = y'(0) = 0$$

$$3) \begin{cases} x' = 4x - 2y - e^t \\ y' = 6x - 3y + e^{-t} \end{cases}$$

$$4) \begin{cases} x_1' = x_1 + x_2 + \sin 2t \\ x_2' = x_1 + x_2 + \cos 2t \end{cases}$$

$$5) \begin{cases} z'' - 3z' + 2z + y' - y = 0 \\ z' - 2z + y' + y = 0 \end{cases}$$

Answer Key

$$1) \quad y = \frac{1}{12}e^{3x} - \frac{2}{3}x^3 - 2c_2e^x + 2c_3e^{-x} + kx + l$$

$$2) \quad z = \frac{1}{2} + \frac{1}{6}e^x - \frac{1}{6}e^{-2x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

$$3) \quad y = 2c_1 + \frac{3}{2}c_2e^t + 6te^t - \frac{3}{2}e^t - \frac{5}{2}e^{-t}$$

$$4) \quad x_1 = c_1 + c_2e^{2t} - \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t, \quad x_2 = -c_1 + c_2e^{2t} + \frac{1}{4}\sin 2t$$

$$5) \quad y = 2c_1 + \frac{1}{2}c_2e^x$$