

# Workbook



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# Matrices

## Matrix and Basic Operations on Matrices

### Questions

1) For each of the following matrix “calculations”, say whether it’s undefined or give the order [size] of the result:

- |   |   |
|---|---|
| a. $A_{4 \times 6} + B_{4 \times 6}$                          | b. $A_{4 \times 6} \cdot B_{4 \times 6}$                  |
| c. $A_{4 \times 6} \cdot C_{6 \times 2} - D_{4 \times 2}$     | d. $A_{4 \times 6} \cdot E_{6 \times 4} - B_{4 \times 6}$ |
| e. $B_{4 \times 6} + A_{4 \times 6} \cdot B_{4 \times 6}$     | f. $E_{6 \times 4} (B_{4 \times 6} + A_{4 \times 6})$     |
| g. $(E_{6 \times 4} + A_{4 \times 6}^T) D_{4 \times 2}$       | h. $E_{6 \times 4}^T \cdot B_{4 \times 6}$                |
| i. $E_{6 \times 4} \cdot A_{4 \times 6} \cdot C_{6 \times 2}$ | j. $E_{6 \times 4} (B_{4 \times 6} - A_{4 \times 6})$     |

2) Solve the matrix equation  $\begin{bmatrix} x+2y & 3x-2y \\ 2x-5y & 2x+8y \end{bmatrix} = \begin{bmatrix} 2-2z & 5+z \\ -4-3z & 3-12z \end{bmatrix}$ , for  $x, y, z$ .

3) Given the matrices  $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ ,  $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$ ,  $E = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 1 & -1 \end{bmatrix}$ .

Evaluate:

- |                        |                    |         |
|------------------------|--------------------|---------|
| a. $E + D$             | b. $E - D + I_3$   | c. $5C$ |
| d. $2D + 4E \cdot I_3$ | e. $2tr(D^2 - 2E)$ |         |

Given the matrices  $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ ,  $D = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 4 & 2 & 10 \end{bmatrix}$ .

- |               |                                    |                          |
|---------------|------------------------------------|--------------------------|
| f. $4C^T + A$ | g. $\frac{1}{2}A^T + \frac{1}{4}C$ | h. $I_2 \cdot B \cdot C$ |
| i. $trC^T C$  | j. $D \cdot A \cdot B \cdot C$     |                          |



4) Rewrite each of the following SLEs in matrix form  $A\underline{x} = \underline{b}$ :

a. 
$$\begin{cases} 2x + y - z = 3 \\ x + 2y - 4z = 5 \\ 6x + 4y + z = 2 \end{cases}$$

b. 
$$\begin{cases} 2x - 3y + z + t = 1 \\ 4x + y + 2z = 4 \\ y + z + t = 1 \\ x - 4z - 2y = 10 \end{cases}$$

5) Given:  $A = \begin{bmatrix} 4 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & -6 & 3 \end{bmatrix}$      $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$      $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .

Express each of the following as an SLE:

a.  $A\underline{x} = \underline{b}$     b.  $A\underline{x} = 4\underline{x} + \underline{b}$     c.  $A\underline{x} = -k\underline{x} + \underline{b}$   
 d.  $A\underline{x} = \underline{x}$     e.  $A^T \underline{x} = 2\underline{x} + 3\underline{b}$

6) Recall that a square matrix  $A$  is *symmetric* if  $A^T = A$  and *antisymmetric* if  $A^T = -A$ .

- Given a square matrix  $A$ , which of the following must be true?
  - $A \cdot A^T$  is symmetric.
  - $A + A^T$  is symmetric.
  - $A - A^T$  is antisymmetric.
- Given 2 antisymmetric  $n \times n$  matrices  $A$  and  $B$ , which of the following must be true:
  - $BABABA$  is antisymmetric
  - $A^2 - B^2$  is symmetric
  - $A^2 + B$  is symmetric
- Given 2 antisymmetric  $n \times n$  matrices  $A$  and  $B$ , which satisfy  $AB = -BA$ . Which of the following must be true:
  - $AB^3$  is antisymmetric
  - $AB^2$  is symmetric
  - $(A - B)^2$  is symmetric
- Given 2 antisymmetric  $n \times n$  matrices  $A$  and  $B$ , which satisfy  $AB = BA$ . Prove:
  - $AB$  is antisymmetric
  - $AB + B$  is antisymmetric
- Given that the  $n \times n$  matrices  $A, B, AB$  are symmetric, prove that  $A^4 B^4 = B^4 A^4$ .

Answer Key

- 1) a.  $C_{4 \times 6}$       b. Undefined.      c.  $B_{4 \times 2}$       d-e. Undefined.  
 f.  $C_{6 \times 6}$       g.  $B_{6 \times 6}$       h. Undefined.      i.  $B_{6 \times 2}$       j.  $C_{6 \times 6}$

2)  $x = 2, y = 1, z = -1$

3) a.  $\begin{bmatrix} 5 & 5 & 3 \\ 0 & 0 & 0 \\ 8 & 3 & 9 \end{bmatrix}_{3 \times 3}$       b.  $\begin{bmatrix} 4 & -3 & -1 \\ -2 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix}_{3 \times 3}$       c.  $\begin{bmatrix} 5 & 20 & 10 \\ 20 & 5 & 25 \end{bmatrix}$

d.  $\begin{bmatrix} 18 & 12 & 8 \\ -2 & 0 & 2 \\ 24 & 8 & 16 \end{bmatrix}$       e. 230      f.  $\begin{bmatrix} 8 & 16 \\ 17 & 6 \\ 7 & 21 \end{bmatrix}_{3 \times 2}$       g.  $\begin{bmatrix} 2\frac{1}{4} & 1\frac{1}{2} & 0 \\ 1 & 1\frac{1}{4} & 1\frac{3}{4} \end{bmatrix}$

h.  $\begin{bmatrix} 8 & 17 & 13 \\ -8 & -2 & -10 \end{bmatrix}_{2 \times 3}$       i. 63      j.  $\begin{bmatrix} -32 & 82 & -22 \\ 48 & 87 & 75 \\ -48 & 108 & -36 \end{bmatrix}$

4) a.  $\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -4 \\ 6 & 4 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}_b$       b.  $\underbrace{\begin{bmatrix} 2 & -3 & 1 & 1 \\ 4 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -2 & -4 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 1 \\ 10 \end{bmatrix}}_b$

5) a.  $\begin{cases} 4x - 2y + 4z = 1 \\ x - y + z = 2 \\ x - 6y + 3z = 3 \end{cases}$       b.  $\begin{cases} -2y + 4z = 1 \\ x - 5y + z = 2 \\ x - 6y - z = 3 \end{cases}$       c.  $\begin{cases} (4+k)x - 2y + 4z = 1 \\ x + (-1+k)y + z = 2 \\ x - 6y + (3+k)z = 3 \end{cases}$

d.  $\begin{cases} 3x - 2y + 4z = 0 \\ x - 2y + z = 0 \\ x - 6y + 2z = 0 \end{cases}$       e.  $\begin{cases} 2x + y + z = 3 \\ -2x - 3y - 6z = 6 \\ 4x + y + z = 9 \end{cases}$

- 6) a. All three are true.      b. (ii) is true.      c. All three are true.  
 d-e. Solution in the recordings.

## Matrix Inverse and its Applications

### Questions

Find the inverse of the matrix:

1)  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2)  $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

3)  $A = \begin{bmatrix} 4 & 1.5 \\ 2 & 1 \end{bmatrix}$

4)  $A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$

5)  $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}$

6)  $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 2 \\ 5 & -3 & 4 \end{bmatrix}$

7)  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

Find the values of  $k$  for which the following matrixes are invertible:

8)  $\begin{bmatrix} 1 & -1 & 1 \\ 5 & -7 & k^2 + 3 \\ 3 & -1 & k + 3 \end{bmatrix}$

9)  $\begin{bmatrix} 1 & 1 & 1 & 1 & k \\ 1 & 1 & 1 & k & 1 \\ 1 & 1 & k & 1 & 1 \\ 1 & k & 1 & 1 & 1 \\ k & 1 & 1 & 1 & 1 \end{bmatrix}$

10) Solve the following SLE by using the Inverse Matrix method:  $\begin{cases} 2x - y + z = 3 \\ 3x - 2y + 2z = 5 \\ 5x - 3y + 4z = 11 \end{cases}$

11) Solve the following SLE by using the Inverse Matrix method:  $\begin{cases} x + 4y + 2z + 4t = 1 \\ x + 2y - z = 0 \\ y + z + t = 1 \\ x + 3y - z - 2t = 0 \end{cases}$



### Answer Key

1)  $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

2)  $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

3)  $\begin{bmatrix} 1 & -1.5 \\ -2 & 4 \end{bmatrix}$

4)  $\begin{bmatrix} -11 & 2 & 2 \\ 4 & -1 & 0 \\ 6 & -1 & -1 \end{bmatrix}$

5)  $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$

6)  $\begin{bmatrix} 2 & -1 & 0 \\ 2 & -3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

7)  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

8)  $k \neq -2, k \neq 1$

9)  $k = 1, k = -4$

10)  $x = 1, y = 2, z = 3$

11)  $x = -13, y = 42, z = -5, t = 2$



## Properties of the Matrix Inverse

### Questions

- 1) Assume that all the matrices are invertible and of order  $n$ , and extract  $X$ :
- a.  $AXC = D$                       b.  $A^{-1}XC = A^{-1}DC$                       c.  $P^{-1}X^T P = A$
- d.  $C^{-1}(A+X)D^{-2} = I$                       e.  $(A-AX)^{-1} = X^{-1}C$                       f.  $ABC^T X^{-1}BA^T C = AB^T$

- 2) Given that  $B = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$  and that  $B^2 X (2B)^{-1} = B + I$ . Find  $X$ .

All matrices are  $2 \times 2$ .

- 3) Given that  $B^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 4 & -1 & 8 \\ 2 & 1 & 3 \end{bmatrix}$  and that  $BYB^T = B^{-1} + B$ . Find  $Y$ .

All matrices are  $3 \times 3$ .

- 4) Given that  $A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$  and that  $5A^T B(I+2A)^{-2} = (7A)^{-2}$ . Find  $B$ .

All matrices are  $2 \times 2$ .

- 5) Prove the following:

- a. Given:  $A$  is a square matrix satisfying  $A^2 - 5A - 2I = 0$ .  
Prove that  $A$  is invertible and express  $A^{-1}$  in terms of  $A$  and  $I$ .
- b. Given:  $A$  is a square matrix satisfying  $(A-3I)(A+2I) = 0$ .  
Prove that  $A$  is invertible and express  $A^{-1}$  in terms of  $A$  and  $I$ .

- 6) Prove the following:

- a. Given:  $A$  is a square matrix satisfying  $A^2 - 5A - 2I = 0$ .  
Prove that  $A$  is invertible and express  $A^{-1}$  in terms of  $A$  and  $I$ .
- b. Given:  $A$  is a square matrix satisfying  $(A-3I)(A+2I) = 0$ .  
Prove that  $A$  is invertible and express  $A^{-1}$  in terms of  $A$  and  $I$ .



- 7) Given:  $A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ -2 & -2 & 6 \end{bmatrix}$ ,  $p(x) = x^3 - 4x^2 - 20x + 48$ .
- Compute  $p(A)$ .
  - Using part a. above, prove that  $A$  is invertible and express  $A^{-1}$  in terms of  $A$  and  $I$ .
- 8) Given:  $A$  is a square matrix satisfying  $A^4 = 0$ .
- Prove that  $A$  is noninvertible.
  - Prove that the matrix  $I - A$  is invertible and find its inverse.
- 9) Given:  $\begin{cases} P^{-1}AP = B \\ Q^{-1}BQ = C \end{cases}$
- Prove that there exists an invertible matrix  $D$  such that  $D^{-1}AD = C$ .

**Answer Key**

- 1) a.  $X = A^{-1}DC^{-1}$       b.  $X = D$       c.  $X = (P^{-1})^T A^T P^T$
- d.  $X = CD^2 - A$       e.  $X = (A + C^{-1})^{-1} A$       f.  $X = BA^T C \cdot (B^T)^{-1} BC^T$
- 2)  $X = \begin{bmatrix} 20 & -4 \\ -8 & 4 \end{bmatrix}$
- 3)  $Y = \begin{bmatrix} 22 & 86 & 38 \\ 64 & 246 & 114 \\ 56 & 222 & 92 \end{bmatrix}$
- 4)  $B = \frac{1}{245} \begin{bmatrix} 264 & 450 \\ 448 & 768 \end{bmatrix}$
- 5) Solution in the recording.
- 6) Solution in the recording.
- 7) a.  $P(A) = 0$       b. Solution in the recording.
- 8) Solution in the recording.
- 9) Solution in the recording.

