

Workbook



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Conservative Vector Fields

Conservative Vector Fields

Questions:

- Determine if \mathbf{F} is a conservative field. If so, find a function ϕ , so that $\nabla\phi = \mathbf{F}$.
 - $\mathbf{F}(x, y) = \langle 6x + 5y, 5x + 4y \rangle$
 - $\mathbf{F}(x, y) = \langle 2x \cos y - y \cos x, -x^2 \sin y - \sin x \rangle$
 - $\mathbf{F}(x, y, z) = z^2 \mathbf{i} + e^{-y} \mathbf{j} + 2xz \mathbf{k}$
 - $\mathbf{F}(x, y, z) = \langle z, 2yz, y^2 \rangle$
- Given the integral $\int_{(1,2)}^{(3,4)} (6xy^2 - y^3) dx + (6x^2y - 3xy^2) dy$
 - Prove that the integral is independent of the path joining $(1,2)$ to $(3,4)$.
 - Compute the integral in two different ways.
- Compute the integral $\int_{(1,4)}^{(3,1)} 2xy^3 dx + (1 + 3x^2y^2) dy$.
- Compute the integral $\int_{(1,0)}^{(2,1)} (2xy - y^4 + 3) dx + (x^2 - 4xy^3) dy$.
- Given the force field $\mathbf{F}(x, y) = e^y \mathbf{i} + xe^y \mathbf{j}$, find the work done by it on a particle which moves from $(1,0)$ to $(-1,0)$ along the curve $y = \sqrt{1-x^2}$.
- Compute the integral $\int_{(1,-1,1)}^{(2,1,-1)} (2xz^3 + 6y) dx + (6x - 2yz) dy + (3x^2z^2 - y^2) dz$.
Give a physical meaning to the result.

7) Given a vector field $\mathbf{F} = \frac{x^2 + y^2 - y}{x^2 + y^2} \cdot \mathbf{i} + \frac{x}{x^2 + y^2} \cdot \mathbf{j}$, and the following three closed curves:

$L_1: x^2 + y^2 = 1$ with positive orientation (counterclockwise).

$L_2: \frac{x^2}{16} + \frac{y^2}{9} = 1$ with negative orientation (clockwise).

$L_3: (x-10)^2 + (y-7)^2 = 1$ with positive orientation (counterclockwise).

Compute:

a. $\oint_{L_1} \mathbf{F} dr$

b. $\oint_{L_2} \mathbf{F} dr$

c. $\oint_{L_3} \mathbf{F} dr$

8) Given a vector field $\mathbf{F} = \frac{-y}{x^2 + y^2} \cdot \mathbf{i} + \frac{x}{x^2 + y^2} \cdot \mathbf{j}$ and the following two paths from

$(2,0)$ to $(-2,0)$: $L_1: x^2 + y^2 = 4, y \geq 0$ $L_2: x^2 + y^2 = 4, y \leq 0$

a. Compute $\int_{L_1} \mathbf{F} dr, \int_{L_2} \mathbf{F} dr$.

b. Prove that \mathbf{F} is conservative in the half-annulus $D = \{(x, y) | 1 \leq x^2 + y^2 \leq 9, y \geq 0\}$.

Remark on Notation:

A vector field has various notations in the technical literature:

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

$$\mathbf{F}(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$$

$$\mathbf{F}(x, y, z) = f(x, y, z)\hat{x} + g(x, y, z)\hat{y} + h(x, y, z)\hat{z}$$

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

Answer Key:

- 1) a. $\phi(x, y) = 3x^2 + 5xy + 2y^2$ b. The field is not conservative
 c. $\phi(x, y) = x^2 \cos y - y \sin x$ d. $\phi(x, y, z) = xz^2 - e^{-y}$
 e. $\phi(x, y, z) = xyz + z^3$ f. The field is not conservative
- 2) a. proved b. 236 3) -58 4) 5 5) -2
- 6) 15 = Work done by force field to move particle from $(1, -1, 1)$ to $(2, 1, -1)$ along curve
- 7) a. 2π b. -2π c. 0
- 8) a. $L_1: \pi, L_2: -\pi$ b. Refer to the videos