



Workbook



Conformal Mappings and Mobius Transformations

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Questions

Conformal Mappings

- 1) Show that the transformation $T(z) = z + 1$ maps the domain $A = \{z \in \mathbb{C} : |z| < 1\}$ to the domain $B = \{w \in \mathbb{C} : |w - 1| < 1\}$.
- 2) Compute $f(-6)$, $f(6)$, $f(6i)$, $f(-6i)$ where $f(z) = e^{\frac{i\pi}{2}} \cdot z$.
What is the geometric meaning of the transformation f ?
- 3) Find the image of the unit circle under the transformation $f(z) = 2z$.
What is the geometric meaning of the transformation f ?
- 4) Find a transformation f which maps the upper half-plane H^+ onto the right half-plane H^r ,
where $H^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ and $H^r = \{z \in \mathbb{C} \mid \text{Re } z > 0\}$.
- 5) Find a transformation f which maps the 1st quadrant Q_1 onto the 3rd quadrant Q_3 ,
where $Q_1 = \{z \in \mathbb{C} \mid \text{Im } z > 0, \text{Re } z > 0\}$ and $Q_3 = \{z \in \mathbb{C} \mid \text{Im } z < 0, \text{Re } z < 0\}$.
- 6) Find the image, B , of the set $A = \{z \in \mathbb{C} : |z| = 2\}$ under the transformation $f(z) = \frac{1}{z}$.
This $f : \mathbb{C} - \{0\} \rightarrow \mathbb{C} - \{0\}$ is known as the *inversion mapping* [transformation].
- 7) Find the image of the set $A = \left\{ z = re^{\frac{i\pi}{4}} : r \geq 0 \right\}$ under the transformation $f(z) = z^2$.

Complex Functions

8) Find the image of the set $A = \{z = e^{i\theta} : 0 \leq \theta \leq \pi\}$ under the transformation $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, known as the Joukowski [Zhukovsky] transformation.

9) Find the image of the set $A = \{z = x + iy : -\pi < y \leq \pi, 0 \leq x \leq 1\}$ under the transformation $f(z) = e^z$.

10) Recall that the principal branch of the logarithm is defined on $\mathbb{C} - (-\infty, 0]$ by

$$\underset{\substack{\text{Log} \\ \text{or Ln}}}{\text{Log}}(z) = \ln r + i\theta \text{ where } z \text{ can be expressed uniquely as } z = re^{i\theta}, r > 0, -\pi < \theta < \pi.$$

Find the image of $\mathbb{C} - (-\infty, 0]$ under the transformation $w = \text{Log } z$.

11) Find the image of $\overline{D(0,1)} - (-\infty, 0]$ under the transformation $w = \text{Log } z$.

Recall: $\overline{D(0,1)} = \{z : |z| \leq 1\}$.

12) Find the image of the square $S = \{z = x + iy : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ under the transformation $w = z^2$. Where is this transformation conformal?

Möbius Transformations

13) Let f, g be Möbius Transformations: $f(z) = \frac{az + b}{cz + d}$, $g(z) = \frac{Az + B}{Cz + D}$.

show that $f \circ g$ is also a Möbius Transformation.

Compute the matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. How is this related to the above?

14) A Möbius Transformation is of the form $f(z) = \frac{az + b}{cz + d}$ with $ad - bc \neq 0$.

Show that if $ad - bc = 0$ then f is a constant function.

15) Find the image of $D = \{z \in \mathbb{C} : |z| < 1\}$ under the mapping $w = f(z) = \frac{z-1}{z+1}$.

16) Find the image of $H^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ under the mapping $w = f(z) = \frac{z-1}{z+1}$.

17) Find the image of $H^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ under the mapping $w = f(z) = \frac{z+i}{z-i}$.

18) Find the image of $D = \{z \in \mathbb{C} : |z| < 1\}$ under the mapping $w = f(z) = \frac{z+i}{z-i}$.

19) Find the image of $H^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ under the mapping $f(z) = \frac{z-a}{z-\bar{a}}$, where $a \in H^+$.

20) Find a conformal mapping* from the open first quadrant QI onto the open unit disk D .

$QI = \{z \in \mathbb{C} : \text{Re } z > 0 \wedge \text{Im } z > 0\}$, $D = \{z \in \mathbb{C} : |z| < 1\}$. *Not necessarily Möbius.

21) Let $f(z) = \frac{z}{z-2i}$ and let $D = \{z \in \mathbb{C} : |z-i| \leq 1 \wedge \text{Re } z \geq 0\}$. Find and sketch $f(D)$.

22) Find a conformal mapping from the slit open unit disk $D(0,1) - [0,1)$ onto the open unit disk D .

23) Find a conformal mapping from the strip $S = \{z \in \mathbb{C} : 0 < \text{Im } z < 1\}$ onto the open first quadrant.

24) Find a 1-1 conformal mapping from $A = \left\{ re^{i\theta} : r > 0 \wedge 0 < \theta < \frac{\pi}{4} \right\}$ onto $B = \{x + iy : 0 < y < 1\}$.

25) Find a 1-1 conformal mapping from $A = \left\{ re^{i\theta} : r > 0 \wedge 0 < \theta < \frac{\pi}{4} \right\}$ onto $B = D(0,1)$ [unit disk].

Complex Functions

26) Find a 1-1 conformal mapping from $A = \mathbb{C} - [0,1]$ [slit plane] onto $B = D(0,1)$ [unit disk].

27) Recall that the upper half plane is $H^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Let $z_1, z_2 \in H^+$.

Find a conformal equivalence* $f : H^+ \rightarrow H^+$ such that $f(z_1) = z_2$.

*conformal equivalence: an analytic map which is 1-1 and onto.

Answer Key

1) [show]

2)

$$f(-6) = -6i$$

$$f(6) = 6i$$

$$f(6i) = i \cdot 6i = -6$$

$$f(-6i) = i(-6i) = 6$$

f is a rotation around 0 by an angle of $\frac{\pi}{2}$.

3)

$\{w \in \mathbb{C} : |w| = 2\}$ (circle with center at 0 and radius 2)

f is a scaling with center 0 and scale factor 2.

4)

$$f(z) = e^{-i\frac{\pi}{2}} \cdot z \quad \text{or} \quad f(z) = -iz$$

5)

$$f(z) = e^{i\pi} \cdot z \quad \text{or} \quad f(z) = -z$$

6)

$$B = \{w \in \mathbb{C} : |w| = \frac{1}{2}\}$$

7) $f(A) = \{w = \rho e^{i\frac{\pi}{2}} : \rho \geq 0\}$

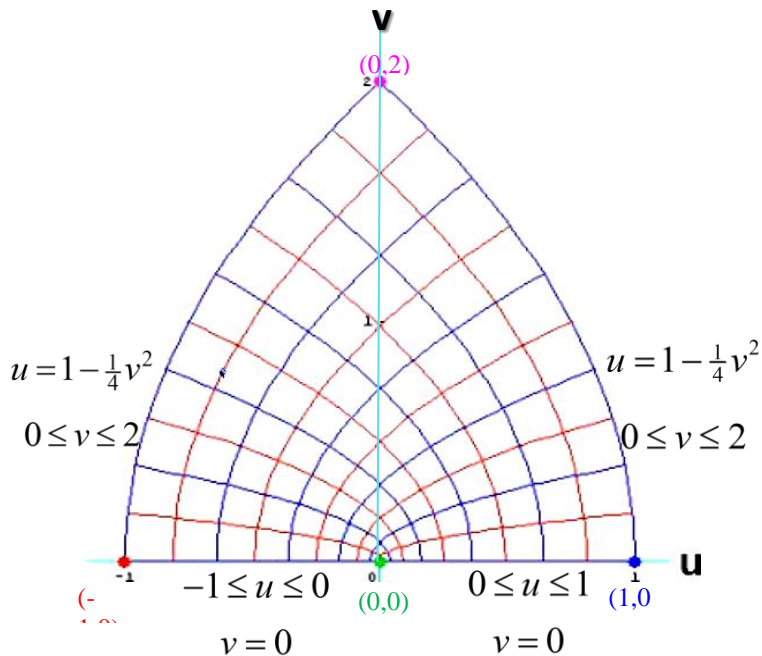
8) $f(A) = [-1, 1] = \{w \in \mathbb{R} : -1 \leq w \leq 1\}$

9) $f(A) = \{w \in \mathbb{C} : 1 \leq |w| \leq e\}$

10) $\{w = u + iv : -\pi < v < \pi\}$

11) $\{w = u + iv : u \leq 0 \wedge -\pi < v < \pi\}$

12)



Not conformal at 0.

13)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} aA + bC & aB + bD \\ cA + bC & cB + bD \end{bmatrix}$$

Matrix of composition = product of the matrices.

14) [show]

15) $f(D) = H^l = \{z : \operatorname{Re} z < 0\}$

16) $f(H^+) = H^+$

17) $f(H^+) = \{z \in \mathbb{C} : |z| > 1\}$ (exterior of closed unit disk).

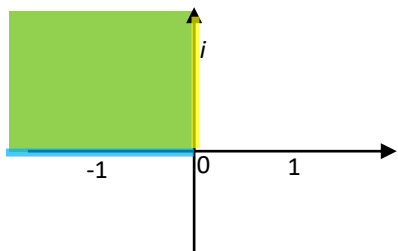
Complex Functions

18) $f(D) = H^l = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$

19) $f(H^+) = D = \{z \in \mathbb{C} : |z| < 1\}$ (open unit disk).

20) $z \mapsto \frac{z^2 - i}{z^2 + i}$ (other answers are possible.)

21) $f(D) = \{z \in \mathbb{C} : \operatorname{Re} z \leq 0, \operatorname{Im} z \geq 0\}$



22)

23) $f(z) = e^{\frac{\pi}{2}z}$ (other answers are possible.)

24) $f(z) = \frac{4}{\pi} \operatorname{Log} z$ (other answers are possible.)

25) $f(z) = \frac{z^4 - i}{z^4 + i}$ (other answers are possible.)

26)

27) $f(z) = \frac{(-z_2 + \overline{z_2})z + (\overline{z_1}z_2 - z_1\overline{z_2})}{-z_1 + z_1}$ (other answers are possible.)