proprep™

Workbook

Complex Series

Numrical Series

Questions

1) Check the convergence of the series $\sum_{n=0}^{\infty}\frac{\cos(in)}{2^n}$.

2) Check the convergence of the series $\sum_{n=0}^{\infty}\frac{n\sin(in)}{3^n}$.

3) Check the convergence of the series $\sum_{n=0}^{\infty}\left(\frac{2+i}{\sqrt{5}}\right)^n$.

4) Check the convergence of the series $\sum_{n=0}^{\infty}\frac{1}{2+i^n}$.

5) Check the convergence of the series $\sum_{n=2}^{\infty}\frac{1}{n^2+i^n}$.

The Cauchy-Hadamard Criterion

6) Find the radius of convergence of the series $\sum_{n=0}^{\infty}e^{in}z^n$.

7) Find the radius of convergence of the series $\sum_{n=0}^{\infty}\left(\frac{z}{in}\right)^n$.

8) Find the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{1}{3^n(2n+1)}(z-3)^n$.

General Series

9) Find the domain of convergence of the series $\sum_{n=0}^{\infty}z^{-n}$.

10) Find the domain of convergence of the series $\sum_{n=0}^{\infty}\frac{z^{-n}}{(1-i)^n}$.

11) Find the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{1}{4^n (z-1)^n}$.

12) Find the domain of convergence of $\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$.

13) Find the domain of convergence of the series $\sum_{n=0}^{\infty} e^{nz}$.

The Weierstrass Test for Uniform Convergence

14) Let $0 < r < 1$ and let $U = \{z \mid |z| \le r\}$. Prove that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly on *U*.

15) Prove that the series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges uniformly on $U = \{z \mid |z| \le 1\}$.

16) Prove that the series $\sum_{n=0}^{\infty} \frac{1}{(z^2-1)^n}$ converges uniformly on $U = \{z \mid |z| \ge 2\}$.

17) Let $0 < r < 1$ and let $U = \{z \mid |z| \le r\}$. Prove that the series $\sum_{n=0}^{\infty} \frac{z^{2n}}{z^n + 1}$ converges uniformly on *U*.

18) Prove that the series $\sum_{n=0}^{\infty} n^{-z}$ converges uniformly on $U = \{z \mid \operatorname{Re} z \ge 2\}$.

Taylor and MacLaurin Series

19) Find the Taylor series for $f(z) = \sin(z+1)$ around $z = 0$.
What is its radius of convergence?

20) Find the Taylor series for $f(z) = \frac{1}{z}$ around $z = i$.
What is its radius of convergence?

21) Find the Taylor series for $f(z) = \frac{2i}{2+i+z}$ around $z = z_0$ where $z_0 \ne -2-i$ is arbitrary.
What is the domain of convergence of the series?

22) Find Taylor series for $f(z) = \frac{1}{(1-z)^3}$ around $z = z_0$ where $z_0 \ne 1$ is arbitrary.
What is the domain of convergence of the series?

Laurent Series

23) Find the Laurent series expansion of the function $f(z)=\frac{1}{1-z}$ $(z \neq 1)$ around $z_0=0$, in
each of the domains where the exists such an expansion.

24) Find the Laurent series expansion of the function $f(z)=\frac{1}{1-\frac{z}{2}}$ $(z \neq 2)$ around $z_0=0$,
in the domain $|z|<2$ and in the domain $|z|>2$. Reminder:

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

25) Find the Laurent series expansion of the function

$$f(z)=\frac{1}{(z+1)(z+3)},\ z \neq -1,-3$$

around $z_0=-1$, in the domain $|z+1|>2$ and in the domain $0<|z+1|<2$.

26) Find the Laurent series expansion of the function

$$f(z)=\frac{1}{(z+2)(z+3)}\ z \neq -2,-3$$

around $z_0=-3$, in each of the domains where the exists such an expansion.

27) Find the Laurent series expansion of the function

$$f(z)=\frac{1}{(z+1)(z+3)},\ z \neq -1,-3$$

around $z_0=0$, in the domain $1<|z|<3$. Hint: use partial fractions.

28) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$ around $z_0=0$,
in the domain $|z|>3$. Hint: use partial fractions.

29) Find the Laurent series expansion of the function $f(z)=\frac{1}{(z+1)(z+3)}$ around $z_0=0$,
in the domain $|z|<1$. Hint: use partial fractions.

30) Find the Laurent series expansion of the function $f(z)=\dfrac{1}{1+z^2}$ $z\neq\pm i$ around $z_0=0$, in the domain $|z|<1$, and find a_{-1}.

31) Find the Laurent series expansion of the function $f(z)=\dfrac{1}{1+z^2}$ $z\neq\pm i$ around $z_0=i$, in the domain $0<|z-i|<2$, and find a_{-1}.

32) Find the Laurent series expansion of the function $f(z)=\dfrac{1}{1+z^2}$ $z\neq\pm i$ around $z_0=i$, in the domain $|z-i|>2$, and find a_{-1}.

33) Find the Laurent series expansion of the function $f(z)=\dfrac{z}{(z-1)(z-4)}$ $z\neq 1,4$ around $z_0=1$, in the domain containing $z=5$.

34) Find the Laurent series expansion of the function $f(z)=\dfrac{1}{(z-2)(z-3)}$ $z\neq 2,3$ around $z_0=0$, in the domain containing $z=1-3i$.

35) Let $f(z)=\dfrac{a}{z-a}$, where $0<a<1$ is a real parameter.

a) Find the Laurent series for *f* around 0 in the domain $|z|>a$.

b)Prove the identity $\sum_{n=1}^{\infty}a^n\cos(n\theta)=\dfrac{a\cos\theta-a^2}{1-2a\cos\theta+a^2}$

36) Let $a\in\mathbb{C}$ and let *f* be analytic on $\mathbb{C}-\{a\}$. Suppose that $\lim_{z\to a}(z-a)f(z)=0$.

Prove that *f* extends uniquely to an entire function $\tilde{f}$ and that

$$\tilde{f}(a)=\frac{1}{2\pi i}\oint_{|z-a|=r}\frac{f(z)}{z-a}dz \quad \forall r>0.$$

37) Let $\sum_{n=-\infty}^{\infty}a_nz^n$ be the Laurent expansion of $\dfrac{z}{e^{z^2}-1}$, $z\neq 0$ around $z_0=0$ in the domain $0<|z|<r$.

a) What is the largest possible value of *r*?

b) Find a_n for all $n\leq 4$

Note: this exercise requires knowledge of the classification of singular points.

38) Compute $\oint\limits_{|z|=1} z^3 \cos\left(\frac{1}{z}\right) dz$ using Laurent series.

Reminder: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, $a_n = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$.

39) Compute $\oint\limits_{|z|=1} e^{\frac{1}{z}} dz$ using Laurent series.

Reminder: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, $a_n = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$.

Answer Key :

1) Diverges

2) Converges

3) Diverges

4) Diverges

5) Converges

6) $R=1$

7) $R=\infty$

8) $R=3$

9) $|z|>1$

10) $|z|>\frac{1}{\sqrt{2}}$

11) $|z-1|>\frac{1}{4}$

12) $2<|z|<4$

13) $\operatorname{Re} z<0$

14)

15)

16)

17)

18)

19) $f(z)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sin 1}{(2n)!}z^{2n}+\frac{\cos 1}{(2n+1)!}z^{2n+1}\right)$

Converges on all $\mathbb{C}$.

20) $f(z)=\frac{1}{i}\cdot\sum_{n=0}^{\infty}i^n(z-i)^n$

Converges on $|z-i|<1$.

21) $f(z)=\frac{1}{i}\cdot\sum_{n=0}^{\infty}i^n(z-i)^n$

Converges on $|z-i|<1$.

$$f(z)=\frac{2i}{2+i+z}$$

$$\frac{1}{1-q}=\sum_{n=0}^{\infty}q^n \qquad |q|<1$$

$$=\frac{2i}{2+i+z_0+(z-z_0)}$$

$$=\frac{2i}{2+i+z_0}\cdot\frac{1}{1+\frac{z-z_0}{2+i+z_0}}$$

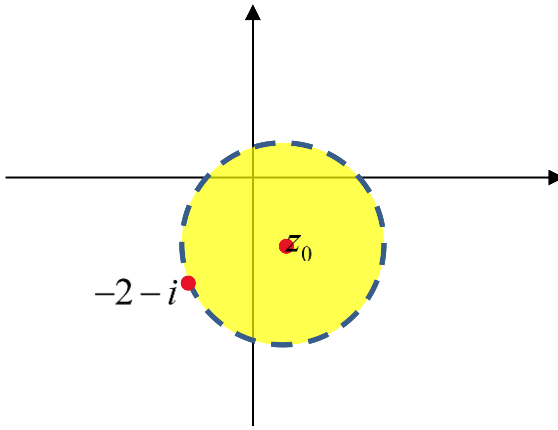
$$=\frac{2i}{2+i+z_0}\cdot\frac{1}{1-\left[-\frac{z-z_0}{2+i+z_0}\right]}$$

$$\frac{1}{1-q}=\sum_{n=0}^{\infty}q^n \qquad |q|<1$$

$$=\frac{2i}{2+i+z_0}\cdot\sum_{n=0}^{\infty}\left[-\frac{z-z_0}{2+i+z_0}\right]^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n 2i}{(2+i+z_0)^{n+1}}\cdot(z-z_0)^n$$

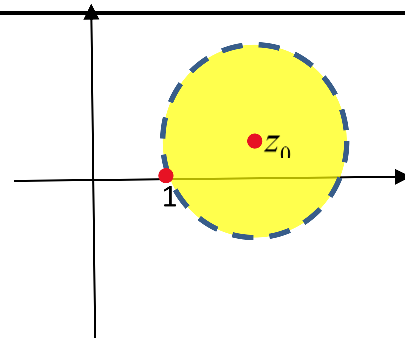
$$\left|-\frac{z-z_0}{2+i+z_0}\right|<1$$



Converges on $|z-z_0|<|2+i+z_0|$.

22) $$\frac{1}{(1-z)^3}=\sum_{n=0}^{\infty}\frac{1}{2}\frac{(n+2)(n+1)}{(1-z_0)^{n+3}}(z-z_0)^n$$

Converges on $|z-z_0|<|1-z_0|$.



23)

$$f(z)=\sum_{n=0}^{\infty}z^n \quad \text{for } |z|<1$$

$$f(z)=\sum_{n=-\infty}^{-1}-z^n \quad \text{for } |z|<1$$

24)

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2^n}z^n \quad \text{for } |z|<2$$

$$f(z)=\sum_{n=-\infty}^{-1}-\frac{1}{2^n}z^n \quad \text{for } |z|>2$$

25)

$$f(z)=\sum_{n=-1}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(z+1)^n \quad \text{for } 0<|z+1|<2$$

$$f(z)=\sum_{n=-\infty}^{-2}\frac{(-1)^n}{2^{n+2}}(z+1)^n \quad \text{for } |z+1|>2$$

26)

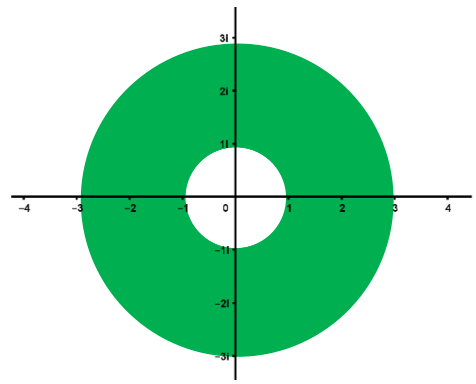
$$f(z)=\sum_{n=-1}^{\infty}-(z+3)^n \ ; \quad 0<|z+3|<1$$

$$f(z)=\sum_{n=-\infty}^{-2}(z+3)^n \ ; \quad |z+3|>1$$

27) Reminder: $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$. Here $z_0=0$.

The plan is to use the geometric series $\boxed{\frac{1}{1-q}=\sum_{n=0}^{\infty} q^n \quad |q|<1}$.

Note that $|z|<3 \Leftrightarrow \left|\frac{z}{3}\right|<1$ and $|z|>1 \Leftrightarrow \left|\frac{1}{z}\right|<1$.



By partial fractions: $f(z)=\frac{1}{2}\frac{1}{z+1}-\frac{1}{2}\frac{1}{z+3}$

$$\frac{1}{2}\frac{1}{z+1}$$

$$=\frac{1}{2z}\cdot\frac{1}{1+\frac{1}{z}}$$

$$=\frac{1}{2z}\cdot\frac{1}{1-\left(-\frac{1}{z}\right)}$$

$$\boxed{\frac{1}{1-q}=\sum_{n=0}^{\infty} q^n \quad |q|<1}$$

$$=\frac{1}{2z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n,\ \left|-\frac{1}{z}\right|<1$$

$$=\frac{1}{2z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n,\ |z|>1$$

$$=\sum_{n=0}^{\infty}\frac{1}{2}\frac{(-1)^n}{z^{n+1}}$$

$$=\sum_{n=0}^{\infty}-\frac{1}{2}\frac{(-1)^{n+1}}{z^{n+1}}$$

$$\frac{1}{2}\frac{1}{z+3}$$

$$=\frac{1}{2}\frac{1}{3-(-z)}$$

$$=\frac{1}{6}\cdot\frac{1}{1-\left(\frac{-z}{3}\right)}$$

$$=\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n,\ \left|\frac{-z}{3}\right|<1$$

$$=\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{-z}{3}\right)^n,\ |z|>3$$

$$=\frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n}$$

$$=\sum_{n=0}^{\infty}\frac{1}{6}\frac{(-1)^n}{3^n}z^n$$

$$=\sum_{n=0}^{\infty}-\frac{1}{2}\frac{(-1)^{n+1}}{z^{n+1}} \qquad =\sum_{n=0}^{\infty}\frac{1}{6}\frac{(-1)^n}{3^n}z^n$$

$$=\sum_{n=1}^{\infty}-\frac{1}{2}\frac{(-1)^n}{z^n}$$

$$=\sum_{n=1}^{\infty}-\frac{1}{2}(-1)^n z^{-n}$$

$$=\sum_{n=-\infty}^{-1}-\frac{1}{2}(-1)^n z^n$$

$$f(z)=\sum_{n=-\infty}^{-1}-\frac{1}{2}(-1)^n z^n-\sum_{n=0}^{\infty}\frac{1}{6}\frac{(-1)^n}{3^n}z^n \quad 1<|z| \text{ and } |z|<3$$

$$\boxed{f(z)=\sum_{n=-\infty}^{-1}-\frac{1}{2}(-1)^n z^n+\sum_{n=0}^{\infty}-\frac{1}{6}\frac{(-1)^n}{3^n}z^n \quad 1<|z|<3}$$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n z^n \quad 1<|z|<3 \text{ where } a_n=\begin{cases}-\frac{1}{2}(-1)^n; & n<0\\ -\frac{1}{6}\frac{(-1)^n}{3^n}; & n\geq 0\end{cases}$$

Partial fractions:

$$\frac{1}{(z+1)(z+3)}=\frac{A}{z+1}+\frac{B}{z+3}$$

$$\frac{1}{(z+1)(z+3)}=\frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$$1=A(z+3)+B(z+1)$$

$$z=-1 \quad\rightarrow\quad A=\frac{1}{2}$$

$$z=-3 \quad\rightarrow\quad B=-\frac{1}{2}$$

$$\frac{1}{(z+1)(z+3)}=\frac{1}{2}\frac{1}{z+1}-\frac{1}{2}\frac{1}{z+3}$$

28) $f(z)=\sum_{n=-\infty}^{-1}(-1)^n\left[-\frac{1}{2}+\frac{1}{6}3^{-n}\right]z^n$

29) $f(z)=\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2}-\frac{1}{6\cdot 3^n}\right]z^n$,

30) $f(z)=\sum_{n=0}^{\infty}(-1)^n z^{2n}$

$a_{-1}=0$

31)

$f(z)=\sum_{n=-1}^{\infty}(-1)^{n+1}\left(\frac{1}{2i}\right)^{n+2}(z-i)^n$

$a_{-1}=\frac{1}{2i}$

32)

$f(z)=\sum_{n=-\infty}^{-2}(-1)^{-n-2}(2i)^{-n-2}(z-i)^n$

$a_{-1}=0$

33) $f(z)=(z-1)^{-1}+\sum_{n=-\infty}^{-2}\frac{4}{9}\cdot 3^{-n}(z-1)^n,\ |z-1|>3$

34) $f(z)=\sum_{n=-\infty}^{-1}(3^{-n-1}-2^{-n-1})z^n,\ |z|>3$

35) a) $f(z)=\sum_{n=-\infty}^{-1}a^{-n}z^n$

b) (proof)

36) (proof)

37)

a) $r_{\max} = \sqrt{2\pi}$

b) $a_{-1} = 1,\ a_1 = -\frac{1}{2},\ a_3 = \frac{1}{12}$

All other a_n are 0.

38) $\oint_{|z|=1} z^3 \cos\left(\frac{1}{z}\right) dz = \frac{\pi i}{12}$

39) $\oint_{|z|=1} e^{\frac{1}{z}} dz = 2\pi i$